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THE UNIQUENESS OF THE SMOOTH SOLUTION IN THE STATIC PROBLEM WITH A COULOMB LAW OF FRICTION AND TWO-SIDED CONTACT†

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The uniqueness of the smooth solution in a problem with friction [1, p. 34] is established for the case when the set of virtual rigid displacements is non-trivial.

1. THE EQUIVALENCE OF THE BOUNDARY-VALUE AND VARIATIONAL FORMULATIONS OF A PROBLEM WITH FRICTION

Let $\Omega \subset R^2$ be a bounded domain with a fairly regular boundary Γ. For the displacement vector $\mathbf{u} = (u_1, u_2)$ we define the strain tensor

$$\varepsilon_{ij} = \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2$$

The stress tensor σ is a linear combination of components of the strain tensor

$$\sigma_{ij} = \sigma_{ij}(\mathbf{u}) = C_{ijpl}\varepsilon_{pl}(\mathbf{u}), \quad p,l = 1,2$$

(summation is understood to take place over repeated indices). The components of the elastic moduli tensor C_{ijpl} possess the usual symmetry properties

$$C_{ijpl} = C_{jilp} = C_{plij}, \quad i,j,p,l = 1,2$$

and we assume the existence of a positive constant C_0 such that

$$C_{ijpl}\varepsilon_{ij}(\mathbf{u}(x))\varepsilon_{pl}(\mathbf{u}(x)) \geq C_0\varepsilon_{ij}(\mathbf{u}(x))\varepsilon_{ij}(\mathbf{u}(x))$$

$$\forall\varepsilon_{ij}(\mathbf{u}(x)), \quad \forall x \in \Omega, \quad i,j,p,l = 1,2$$

We consider the formulation of the problem with friction.

Suppose that on a part Γ_1 of its boundary Γ the body Ω is subjected to the action of surface forces

$$p_i = \sigma_i = \sigma_{ij}n_j, \quad i = 1,2$$

(where $\mathbf{n} = (n_1, n_2)$ is the unit vector of the outward normal to Γ). On the part Γ_0 the following boundary conditions are given

$$u_n = \mathbf{u} \cdot \mathbf{n} = 0, \quad \sigma_t = 0$$

where σ_t is the shear stress vector, and on the part Γ_f of the boundary Γ boundary conditions are imposed which correspond to the Coulomb law of friction and two-sided contact [1]

$$\sigma_n = \sigma_{ij}n_jn_i = T_n, \quad T_n = \mathbf{T} \cdot \mathbf{n}$$

if $|\sigma_t| < g$, then $\mathbf{u}_t = 0$ ($g > 0$ on Γ_f); if $|\sigma_t| = g$, then a $\lambda \geqslant 0$ exists such that $u_t = -\lambda\sigma_t$.

Here $\mathbf{T} = (T_1, T_2)$ is the force with which the second body acts on Ω in the contact zone, $g(x)$ is the value of the friction force, and $\mathbf{u}_t$ is the shear component of u.

In the domain Ω we have the equilibrium condition

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$$-\partial[\sigma_{ij}(\mathbf{u})]/\partial x_j = F_i,\quad i = 1,2 \tag{1.1}$$

where $\mathbf{F} = (F_1, F_2)$ is the vector of forces specified in Ω.

Below we shall use the functional spaces [2] $L_2\Omega$, $L_2(\Gamma)$, $W_2^1(\Omega)$, $W_2^{1/2}(\Gamma)$, $W_2^2(\Omega)$.

We introduce the Hilbert space [3, p. 27]

$$H = \{\upsilon \in [W_2^1(\Omega)]^2,\ \ \upsilon_n = 0 \ \text{ on } \Gamma_0\}$$

with a scalar product induced from $[W^1{}_2(\Omega)]^2$.

We consider the extremal (variational) problem [1, 4]

$$J(\mathbf{u}) = \frac{1}{2}a(\mathbf{u},\mathbf{u}) - L(\mathbf{u}) + j(\mathbf{u}) \to \min,\quad \mathbf{u} \in H \tag{1.2}$$

where

$$\alpha(\mathbf{u},\upsilon) = \int_\Omega C_{ijpl}\varepsilon_{ij}(\mathbf{u})\varepsilon_{pl}(\upsilon)d\Omega$$

$$L(\mathbf{u}) = \int_\Omega F_i u_i d\Omega + \int_{\Gamma_f} T_n u_n d\Gamma + \int_{\Gamma_1} P_i u_i d\Gamma,\quad j(u) = \int_{\Gamma_f} g|\mathbf{u}_t|d\Gamma.$$

$$\mathbf{F} = (F_1, F_2) \in [L_2(\Omega)]^2,\quad \mathbf{P} = (P_1, P_2) \in [L_2(\Gamma_1)]^2$$

$$\mathbf{T} = (T_1, T_2) \in [L_2(\Gamma_f)]^2,\quad g(x) \in L_\infty(\Gamma_f),\quad g > 0 \ \text{ on } \Gamma_f$$

The solution of the problem with friction is also the solution of problem (1.2). If the solution of problem (1.2) belongs to the space $[W^2{}_2(\Omega)]^2$, then it is also the solution of the problem with friction. These facts are proved in the same way as in [1], where the problem with friction is investigated with the condition $\Gamma_0 = \Gamma_1 = \phi$.

In fact, suppose that u is a solution of the problem with friction. It follows from the boundary conditions of the problem that

$$\sigma_t(\mathbf{u})\cdot\mathbf{u}_t + g|\mathbf{u}_t| = 0 \ \text{ on } \Gamma_f$$

and so for any $\upsilon \in H$ we have

$$\sigma_t(\mathbf{u})\cdot(\upsilon_t - \mathbf{u}_t) + g(|\mathbf{v}_t| - |\mathbf{u}_t|) \geq 0 \ \text{ on } \Gamma_f \tag{1.3}$$

We multiply both sides of Eq. (1.1) by $\upsilon - \mathbf{u}$ and, using Green's formula, we obtain

$$\begin{aligned} a(\mathbf{u},\upsilon - \mathbf{u}) - &\int_{\Gamma_f\cup\Gamma_0} [\sigma_t(u)\cdot(\upsilon_t - \mathbf{u}_t) + \sigma_n(u)(\upsilon_n - u_n)]d\Gamma - \\ &- \int_{\Gamma_1} \sigma_i(u)(\upsilon_i - u_i)]d\Gamma = \int_\Omega F_i(\upsilon - u)_i d\Omega \end{aligned} \tag{1.4}$$

It follows from the boundary conditions that

$$\int_{\Gamma_0}[\sigma_t(u)\cdot(\upsilon_t - \mathbf{u}_i) + \sigma_n(u)\upsilon_n - u_n)]d\Gamma = 0$$

Hence

$$a(\mathbf{u},\upsilon - \mathbf{u}) - L(\upsilon - \mathbf{u}) = \int_{\Gamma_f} \sigma_t(u)\cdot(\upsilon_t - \mathbf{u}_t)d\Gamma$$

We put $j(\upsilon) = \int_{\Gamma_f} g|\mathbf{u}_t|d\Gamma$ and add to both sides of the equation the expression $j(\upsilon) - j(\mathbf{u})$. We obtain (using (1.3))

$$a(\mathbf{u},\upsilon - \mathbf{u}) + j(\upsilon) - j(\mathbf{u}) - L(\upsilon - \mathbf{u}) = I_1 \geq 0$$

$$I_1 = \int_{\Gamma_f} [\sigma_t(\mathbf{u})\cdot(\upsilon_t - \mathbf{u}_t) + g(|\upsilon_t| - |\mathbf{u}_t|)]d\Gamma$$

from which the variational inequality

$$a(\mathbf{u},\upsilon - \mathbf{u}) + j(\upsilon) - j(\mathbf{u}) \geq L(\upsilon - \mathbf{u}),\quad \forall\upsilon \in H \tag{1.5}$$

follows, which is equivalent to problem (1.1) (see [5]).

Suppose now that $\mathbf{u} \in [W^2{}_2(\Omega)]^2$ is a solution of problem (1.2), and, of course, also a solution of inequality (1.5) and $D(\Omega)$ is the space of infinitely differentiable functions with compact support in Ω. In the variational inequality (1.5) we put $\upsilon = \mathbf{u} \pm \varphi$, $\varphi \in [D(\Omega)]^2$. Bearing in mind that all boundary integrals are equal to zero and $j(\mathbf{u} \pm \varphi) - j(\mathbf{u}) = 0$, we have

$$a(\mathbf{u},\boldsymbol{\varphi})=\int_\Omega F_i\varphi_i d\Omega\ ,\quad \forall\Omega\in[D(\Omega)]^2$$

from which (using the smoothness of the solution) the equilibrium equation (1.1) follows.

Then, applying Green's formula, we derive Eq. (1.4) from the equilibrium equations. Together with the variational inequality this gives

$$\int_{\Gamma_1}(\sigma_i(u)-P_i)(\upsilon_i-\kappa_i)d\Gamma+I_1+I_2\geq 0,\quad \forall\upsilon\in H \tag{1.6}$$

where

$$I_2=\int_{\Gamma_f}(\sigma_n(u)-T_n)(\upsilon_n-u_n)d\Gamma.$$

Let Ψ_0 be the space of vector functions $\psi\in[W_2^{1/2}(\Gamma)]^2$ with supports in Γ_1. We take $\upsilon=\mathbf{u}\pm\psi$, where $\psi\in\Psi_0$. It is clear that in this case

$$\int_{\Gamma_1}(\sigma_i(u)-P_i)\psi_i d\Gamma=0,\quad \forall\psi\in\Psi_0$$

From this it follows [1] that $\sigma_1(u)=P_1$ almost everywhere in Γ_1. This also gives $I_1+I_2\geqslant 0$, $\forall\upsilon\in H$.

We put $\upsilon=\mathbf{u}\pm\varphi$, where $\varphi\in[W_2^{1/2}(\Gamma)]^2$ is a vector function such that $|\varphi_t|=0$ on Γ_f, and $\varphi_n\in W_2^{1/2}(\Gamma)$ is an arbitrary function with support in Γ_f. We then have $\sigma_n=T_n$ almost everywhere on Γ_f [1]. It then follows from (1.6) that

$$I_1\geq 0,\ \forall\upsilon\in H. \tag{1.7}$$

Let Ψ be the space of vector functions $\psi\in[W_2^{1/2}(\Gamma)]^2$ with support in Γ_f. If $\psi\in\Psi$ then we expand ψ as follows: $\psi=\psi_n\mathbf{n}+\psi_t$, ψ_t, $\psi_n=\psi\cdot\mathbf{n}$ and put $\upsilon_t=\psi_t$ in (1.7). Since $\sigma_t(\upsilon)\cdot\mathbf{n}=0$ and $\upsilon\in H$, we have $\sigma_t(\upsilon)\cdot\psi_t=\sigma_t(\upsilon)\cdot\psi$. Using $|\psi_t|\leqslant|\psi|$, we obtain

$$\int_{\Gamma_f}[\sigma_t(u)\cdot\psi+g|\psi|]d\Gamma-\int_{\Gamma_f}[\sigma_t(\mathbf{u})\cdot\mathbf{u}_t+g|\mathbf{u}_t|]d\Gamma\geq 0,\quad \forall\psi\in\Psi$$

from (1.7).

Replacing ψ by $\pm\lambda\psi$ ($\lambda\geqslant 0$) we find that

$$\left|\int_{\Gamma_f}\sigma_t(u)\cdot\psi d\Gamma\right|\leq\int_{\Gamma_f}g|\psi|]d\Gamma,\quad \forall\psi\in\Psi;\quad \int_{\Gamma_f}[\sigma_t(\mathbf{u})\mathbf{u}_t+g|\mathbf{u}_t|]d\Gamma\leq 0 \tag{1.8}$$

In the first inequality in (1.8) we represent $\sigma_t(u)$ by $g^{-1}\sigma_\tau(u)g$. We then obtain [1, p. 136] $|\sigma_t(\mathbf{u})|\leqslant g$ almost everywhere on Γ_f. In this case $\sigma_t(\mathbf{u})\cdot\mathbf{u}_t+g|\mathbf{u}_t|\geqslant 0$, which, together with the second inequality (1.8), gives the equality

$$\sigma_t(\mathbf{u})\cdot\mathbf{u}_t+g|\mathbf{u}_t|=0 \text{ a.e. on } \Gamma_f \tag{1.9}$$

equivalent to the boundary condition which is imposed on the shear components $\mathbf{u}_t$ and σ_t in the problem with friction.

Hence problem (1.2) can be considered as a variational formulation of the original problem with friction.

2. THE UNIQUENESS OF THE SOLUTION OF THE PROBLEM WITH FRICTION

The kernel R of the bilinear form $a(\mathbf{u},\upsilon)=\int_\Omega C_{ijpl}\varepsilon_{ij}(\upsilon)d\Omega$ consists of the vector function $\rho=(\rho_1,\rho_2)$ where $\rho_1(x)=a_1-bx_2$, $\rho_2(x)=a_2+bx_1$ and a_1, a_2 and b are arbitrary fixed numbers. The subspace $\tilde{R}=H\cap R$ is a set of virtual rigid displacements (i.e. displacements of Ω as an absolutely rigid body, retaining the strict restrictions).

If for any non-zero vector $\rho\in\tilde{R}$ we have the inequality

$$\int_{\Gamma_f}g|\rho_t|d\Gamma-|L(\rho)|>0 \tag{2.1}$$

then the variational problem (1.2) is solvable.

The condition of solvability is proved in the same way as in [1], where a similar condition was derived for the case $\Gamma_f=\Gamma$. Indeed, because the functional J is continuous and convex, it is sufficient to show [1, p. 68] that

$$J(\mathbf{u})\to+\infty,\ \text{if}\ \|\mathbf{u}\|\to\infty,\ u\in H \tag{2.2}$$

For $\upsilon \in H$ we put $\mathbf{w} = \upsilon - Q\upsilon$, where Q is the orthogonal projection in $[L_2(\Omega)]^2$ from H onto $\tilde{R}$. Hence $\upsilon = \mathbf{w} + \rho$, $\rho \in \tilde{R}$. We put $\varepsilon(\upsilon) = \int_\Omega \varepsilon_{ij}(\upsilon)\varepsilon_{ij}(\upsilon)d\Omega$ and $\|\cdot\|_0$ is the norm in $[L_2(\Omega)]^2$.

We have [5, p. 80]

$$a(\mathbf{w}, \mathbf{w}) \geqslant C_0\varepsilon(\mathbf{w}, \mathbf{w}) \geqslant C_1\int_\Omega \frac{\partial w_i}{\partial w_k}\frac{\partial w_i}{\partial w_k}d\Omega \geqslant C_2\left\|\mathbf{w}\right\|^2$$

where C_0, C_1 and C_2 are positive constants.

Furthermore

$$\varepsilon(\upsilon)+\|\upsilon\|_0^2 = \varepsilon(\mathbf{w})+\|\mathbf{w}\|_0^2+\|\rho\|_0^2 \tag{2.3}$$

By the Korn inequality expression $\|\upsilon\| = (\varepsilon(\upsilon) + \|\upsilon\|_0^2)^{1/2}$ is a norm in H. From (2.3) the norm $\|\upsilon\|$ is equivalent to $\|\mathbf{w}\| + \|\rho\|_0$. From (2.1) and the finite dimension of the set $\tilde{R}$ a constant $C > 0$ exists such that

$$\int_{\Gamma_f} g|\rho_t|d\Gamma - |L(\rho)| \geq C\|\rho\|_0$$

Moreover

$$|j(\mathbf{w}+\rho)-j(\rho)| \leq \left|\int_{\Gamma_f} g|\mathbf{w}_t|d\Gamma\right| \leq C_3\|\mathbf{w}\|_0 \quad (C_3 = \text{const} > 0)$$

Hence

$$J(\upsilon) \geq \frac{1}{2}a(\mathbf{w},\mathbf{w})+j(\mathbf{w}+\rho)+C\|\rho\|_0-j(\rho)-L(\mathbf{w}) \geq C_4(\|\mathbf{w}\|^2+\|\rho\|_0)+C_5\|\mathbf{w}\|$$

(where C_4 and C_5 are positive constants).

Using the fact that $\|\upsilon\|$ is equivalent to the quantity $\|\mathbf{w}\| + \|\rho\|_0$ assertion (2.2) is proved.

Condition (2.1) guarantees the existence of a solution of the variational problem (1.2). The question of the uniqueness of the solution of the problem with friction in the case when $\tilde{R}$ is non-trivial appears not to have been investigated [1, 6] up until now.

Below we shall take Γ_0 and Γ_f to be rectilinear sections. For convenience we will make the x_1 coordinate axis lie along Γ_0. Then the subspace $\tilde{R} = H \cap R$ consists of vector functions of the form $\rho = (a, 0)$, where a is an arbitrary constant.

Note that in order to satisfy the condition of solvability (2.1) it is necessary for the sections Γ_f and Γ_0 to be non-orthogonal, as otherwise $j(\rho) = \int_{\Gamma_f} g|\rho_t|d\Gamma = 0$ and inequality (2.1) is not satisfied.

Given these assumptions about Γ_0 and Γ_f the following theorem holds.

Theorem. Suppose that the condition of solvability (2.1) is satisfied. Then the solution of variational problem (1.2) is unique in the space $[W_2^2(\Omega)]^2$.

Proof. Suppose there are two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ in $[W_2^2(\Omega)]^2$. We have

$$0 = J(\mathbf{u}_1+(\mathbf{u}_2-\mathbf{u}_1))-J(\mathbf{u}_1) = a(\mathbf{u}_1,\mathbf{u}_2-\mathbf{u}_1)-L(\mathbf{u}_2-\mathbf{u}_1)+j(\mathbf{u}_2)-$$
$$-j(\mathbf{u}_1)+\frac{1}{2}a(\mathbf{u}_2-\mathbf{u}_1,\mathbf{u}_2-\mathbf{u}_1)$$

In view of (1.5) the last term on the right-hand side of this equation vanishes. Therefore $\mathbf{u}_2 - \mathbf{u}_1 \in \tilde{R}$, i.e. $\mathbf{u}_2 = \mathbf{u}_1 + \rho$ where $\rho = (a, 0)$.

We put $\tilde{\Gamma}_i = \{x \in \Gamma_f : |\sigma_t(\mathbf{u}_i)| < g\}$ $(i = 1, 2)$. It is clear that $\tilde{\Gamma}_1 = \tilde{\Gamma}_2$. We consider the two possible cases: (1) mes $\tilde{\Gamma}_1 > 0$, and (2) mes $\tilde{\Gamma}_1 = 0$. In the first case it follows from (1.9) that

$$|(\mathbf{u}_1)_t| = |(\mathbf{u}_2)_t| = 0 \text{ on } \tilde{\Gamma}_1.$$

But $(\mathbf{u}_2)_t = (\mathbf{u}_1)_t + \rho$, from which $\rho_t = 0$ on $\tilde{\Gamma}_1$. Since the set Γ_f is not orthogonal to Γ_0, $\rho = (0, 0)$ and therefore $\mathbf{u}_1 = \mathbf{u}_2$.

Suppose now that mes $\tilde{\Gamma}_1 = 0$, i.e. $|\sigma_t| = g$ almost everywhere on Γ_f. We have

$$J(u_1) = \frac{1}{2}a(\mathbf{u}_1,\mathbf{u}_1)-L(\mathbf{u}_1)+j(\mathbf{u}_1)$$

$$J(u_1+\rho) = \frac{1}{2}a(\mathbf{u}_1,\mathbf{u}_1)-L(\mathbf{u}_2)+j(\mathbf{u}_2)$$

Subtracting the second equation from the first, we obtain

$$L(\mathbf{u}_2-\mathbf{u}_1)+j(\mathbf{u}_1)-j(\mathbf{u}_2)=0, \quad L(\rho)+j(\mathbf{u}_1)-j(\mathbf{u}_2)=0 \tag{2.4}$$

Then, using (1.9), we have

$$\rho_t\sigma_t(\mathbf{u}_1)+g\zeta=0, \quad |\rho_t||\sigma_t(\mathbf{u}_1)|=g|\zeta|, \quad |\rho_t|g=g|\zeta|$$
$$|\rho_t|=|\zeta| \quad (\zeta=|(\mathbf{u}_1)_t+\rho_t|-|(\mathbf{u}_1)_t|)$$

almost everywhere on Γ_f.

Since Γ_f is a rectilinear section and $\rho=(a,0)$, we have $|\rho_t|=\tilde{c}=\text{const}\neq 0$ on Γ_f (the set Γ_0 being non-orthogonal to Γ_f). We know [2, p. 50] that the space $W_2^2(\Omega)$ is contained in the space $C(\bar{\Omega})$ of functions continuous on Ω. Hence the function ζ is continuous on Γ_f. We put

$$\Gamma_{\tilde{c}}=\{x\in\Gamma_f:\zeta=\tilde{c}\}, \quad \Gamma_{-\tilde{c}}=\{x\in\Gamma_f:\zeta=-\tilde{c}\}$$

It is clear that mes $(\Gamma_{\tilde{c}}\cup\Gamma_{-\tilde{c}})=$ mes Γ_f. Suppose that mes $\Gamma_{\tilde{c}}>0$ and mes $\Gamma_{-\tilde{c}}>0$ both hold. Then it follows from the continuity of ζ on Γ_f that a point x_0 exists at which $\zeta=0$. Thus a $\delta>0$ exists such that for all $x\in B_\delta(x_0)$ $\cap\,\Gamma$ ($B_\delta(x_0)=\{x\in R^2\colon |x-x_0|<\delta|\}$) the inequality $\zeta<\tilde{c}/2$ is satisfied. Since mes $(B_\delta(x_0)\cap\Gamma)>0$, we obtain a contradiction to the condition mes $(\Gamma_{\tilde{c}}\cup\Gamma_{-\tilde{c}})$. Hence either mes $\Gamma_{\tilde{c}}=$ mes Γ_f or mes $\Gamma_{-\tilde{c}}=$ mes Γ_f.

If mes $\tilde{\Gamma}_c=$ mes Γ_f, it follows from (2.4) that

$$L(\rho)+\int_{\Gamma_f} g|\rho_t|\,d\Gamma=0$$

which contradicts the condition of solvability (2.1).

If, however, mes $\Gamma_{-\tilde{c}}=$ mes Γ_f, then from (2.4) we have

$$L(\rho)-\int_{\Gamma} g|\rho_t|\,d\Gamma=0$$

which again contradicts the condition of solvability (2.1). The theorem is proved.

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